NEW REFINEMENTS OF NARAYANA POLYNOMIALS AND MOTZKIN POLYNOMIALS

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ABSTRACT. Chen, Deutsch and Elizalde introduced a refinement of the Narayana polynomials by distinguishing between old (leftmost child) and young leaves of plane trees. They also provided a refinement of Coker's formula by constructing a bijection. In fact, Coker's formula establishes a connection between the Narayana polynomials and the Motzkin polynomials, which implies the γ-positivity of the Narayana polynomials. In this paper, we introduce the polynomial $G_n(x_{11}, x_{12}, x_2; y_{11}, y_{12}, y_2)$, which further refines the Narayana polynomials by considering leaves of plane trees that have no siblings. We obtain the generating function for $G_n(x_{11}, x_{12}, x_2; y_{11}, y_{12}, y_2)$. To achieve further refinement of Coker's formula based on the polynomial $G_n(x_{11}, x_{12}, x_2; y_{11}, y_{12}, y_2)$, we consider a refinement $M_n(u_1, u_2, u_3; v_1, v_2)$ of the Motzkin polynomials by classifying the old leaves of a tip-augmented plane tree into three categories and the young leaves into two categories. The generating function for $M_n(u_1, u_2, u_3; v_1, v_2)$ is also established, and the refinement of Coker's formula is immediately derived by combining the generating function for $G_n(x_{11}, x_{12}, x_2; y_{11}, y_{12}, y_2)$ and the generating function for $M_n(u_1, u_2, u_3; v_1, v_2)$. We derive several interesting consequences from this refinement of Coker's formula, including the new symmetries of vertices of plane trees, Euler transformation of the Narayana polynomials and the Motzkin polynomials due to Lin and Kim and the real-rootedness of the Motzkin polynomials. The method used in this paper is the grammatical approach introduced by Chen. We develop a unified grammatical approach to exploring polynomials associated with the statistics defined on plane trees. The derivations of the generating functions for $G_n(x_{11}, x_{12}, x_2; y_{11}, y_{12}, y_2)$ and $M_n(u_1, u_2, u_3; v_1, v_2)$ become quite simple once their grammars are established.

1. INTRODUCTION

The main objective of this paper is to present a grammatical approach to studying the Catalan numbers, the Narayana numbers, the Motzkin numbers and related topics. The Catalan numbers and the Motzkin numbers are classical topics in enumerative combinatorics, which have been extensively investigated for decades, see [20, 21, 31, 36].

The Narayana numbers can be viewed as a refinement of the Catalan numbers. It is well-known that the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ counts the number of unlabeled

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(rooted) plane trees with n edges. In this context, the Narayana number $N(n,k)$ provides a more detailed enumeration by counting the number of such plane trees with n edges and k leaves. The explicit formula for $N(n,k)$ is given by

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1} \tag{1.1}$$

with $N(0,0) = 1$ and $N(0,k) = 0$ for $k \geq 1$. The Narayana polynomials are defined as

$$N_n(x) = \sum_{k=1}^{n} N(n,k)x^k$$

and the generating function for the Narayana polynomial $N_n(x)$ is given by

$$\sum_{n=0}^{\infty} N_n(x)q^n = \frac{1+q-xq-\sqrt{1-2(1+x)q+(x-1)^2q^2}}{2q}, \tag{1.2}$$

see Deutsch [19] and Petersen [31, p. 24].

Let $\mathcal{P}_n$ denote the set of unlabeled (rooted) plane trees with n edges. For a plane tree $T \in \mathcal{P}_n$, let $\mathrm{leaf}(T)$ denote the number of leaves in T. The Narayana polynomials can be expressed as

$$N_n(x) = \sum_{T\in\mathcal{P}_n} x^{\mathrm{leaf}(T)}. \tag{1.3}$$

For example, as illustrated by the plane trees in Fig. 1, we observe that $N_3(x) = x + 3x^2 + x^3$.

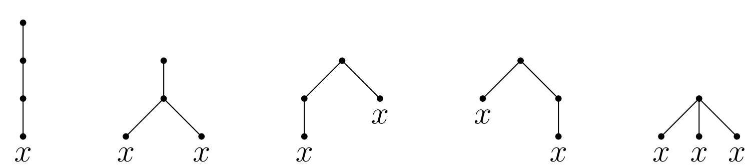


FIGURE 1. 3-edge plane trees.

The Narayana polynomials have been extensively studied, see, for example, Bonin, Shapiro and Simion [1], Coker [18], Mansour and Sun [28, 29], Rogers [32], Rogers and Shapiro [33] and Sulank [37,39]. In particular, the Narayana polynomial $N_n(x)$ is also the h-polynomial of the simplicial complex dual to an associahedron of type A_n, see Formin and Reading [23].

Using (1.1), we see that $N(n,k) = N(n,n-k+1)$ for $1 \leq k \leq n$, leading to the following combinatorial consequence. Elegant combinatorial proofs of this proposition were provided by Chen [4, 6] and Schmitt and Waterman [34].

Proposition 1.1. *The number of plane trees with n edges and k leaves is equal to the number of plane trees with n edges and k interior vertices.*

It follows that $N_n(x)$ is a symmetric polynomial. Recall that a polynomial $a_0 + a_1 q + \cdots + a_N q^N$ with integer coefficients is called symmetric if $a_j = a_{N-j}$ for $0 \le j \le N$. It is known that the Narayana polynomial $N_n(x)$ has only distinct non-positive real zeros, see Petersen [31, Problem 4.7]. As pointed out by Brändén [2, Remark 7.3.1], the real-rootedness of a polynomial with symmetric and non-negative coefficients implies the γ-positivity of such polynomial. In fact, the γ-positivity of the Narayana polynomial $N_n(x)$ can be deduced from the following formula due to Coker [18].

Theorem 1.1 (Coker)**.** *For $n \ge 1$,*

$$N_n(x) = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-1}{2k-2} C_{k-1} x^k (1+x)^{n-2k+1}, \tag{1.4}$$

where C_k is the k-th Catalan number.

Coker [18] obtained this formula by using the Lagrange inversion formula. Chen, Yan and Yang [16] provided a bijective proof of this formula by building a bijection between Dyck paths and 2-Motzkin paths. By comparing the coefficients of x^n in (1.4), one can recover an identity of Simion and Ulman [35] expressing the Narayana numbers by the Catalan numbers. The bijective proof of this identity was given by Chen, Deng and Du [7].

Let $P_e(n)$ ($P_o(n)$) denote the number of plane trees with n edges and an even (odd) number of leaves. Setting $x = -1$ in (1.4) and using (1.3), we obtain

Corollary 1.1. *For $n \ge 1$,*

$$P_e(2n) - P_o(2n) = 0,$$

$$P_e(2n+1) - P_o(2n+1) = (-1)^{n+1} C_n.$$

These relations were also obtained by Bonin, Shapiro and Simion [1], Eu, Liu and Yeh [22] and Klazar [24] independently. A bijective proof of these two relations was given by Chen, Shapiro and Yang [15].

In fact, Coker's formula (1.4) built a connection between the Narayana numbers and the Motzkin numbers, see Chen and Pan [17]. The Motzkin numbers, introduced in [30] and denoted M_n, are defined by the following recurrences:

$$M_0 = 1, \quad M_1 = 1, \quad M_{n+1} = M_n + \sum_{k=0}^{n-1} M_k M_{n-1-k} \quad \text{for } n \ge 1.$$

The following two identities describe the relationships between the Motzkin numbers M_n and the Catalan numbers C_n:

$$(a)\ M_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C_k \quad \text{and} \quad (b)\ C_{n+1} = \sum_{k=0}^{n} \binom{n}{k} M_k. \tag{1.5}$$

Combinatorial proofs of the above two relations were given by Donaghey [20].

Chen and Pan [17] defined the following two-variable Motzkin polynomials:

$$M_n(u;v)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}C_k u^{k+1}v^{n-2k}. \tag{1.6}$$

It should be noted that when u and v are positive integers, the Motzkin polynomial $M_n(u;v)$ was termed the generalized Motzkin numbers by Sun [40]. Sun [40] also obtained the generating function for $M_n(u;v)$:

Theorem 1.2 (Sun)**.**

$$\sum_{n\geq 0}M_n(u;v)q^n=\frac{1-vq-\sqrt{1-2vq+(v^2-4u)q^2}}{2q^2}. \tag{1.7}$$

In the notation of $N_n(x)$ and $M_n(u;v)$, we find that Coker's result (1.4) can be recast as follows:

Theorem 1.3 (Coker)**.** *For* $n\geq 0$,

$$N_{n+1}(x)=M_n(x;1+x). \tag{1.8}$$

In fact, Theorem 1.3 follows immediately from comparing the generating function (1.2) for $N_n(x)$ with the generating function (1.7) for $M_n(u;v)$.

Chen, Deutsch and Elizalde [8] refined the Narayana polynomials by classifying the leaves of a plane tree into old and young leaves. Specifically, a leaf is considered an old leaf if it is the leftmost child of its parent, and a young leaf otherwise. It should be noted that if a leaf is the only child of its parent, Chen, Deutsch and Elizalde also classified this leaf as an old leaf.

Let $\mathrm{oleaf}(T)$ and $\mathrm{yleaf}(T)$ denote the numbers of old leaves and young leaves in a plane tree T, respectively. Chen, Deutsch and Elizalde [8] defined the following polynomials:

$$G_n(x_1,x_2)=\sum_{T\in\mathcal{P}_n}x_1^{\mathrm{oleaf}(T)}x_2^{\mathrm{yleaf}(T)}. \tag{1.9}$$

When $x_1=x_2=x$ in $G_n(x_1,x_2)$, we recover the Narayana polynomial $N_n(x)$.

Fig. 2 shows five plane trees with three edges, where the old leaves are labeled by x_1 and the young leaves are labeled by x_2. Hence, we have

$$G_3(x_1,x_2)=x_1+2x_1x_2+x_1^2+x_1x_2^2.$$

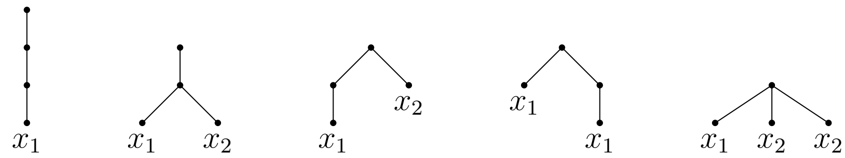


FIGURE 2. 3-edge plane trees.

Chen, Deutsch and Elizalde [8] derived the following generating function for $G_n(x_1, x_2)$ by establishing an equation for the generating function using a decomposition of plane trees.

Theorem 1.4 (Chen-Deutsch-Elizalde)**.**

$$\sum_{n\geq 0} G_n(x_1,x_2)q^n = \frac{1+(1-x_2)q-\sqrt{1-2(1+x_2)q+(1-4x_1+2x_2+x_2^2)q^2}}{2q}. \tag{1.10}$$

Let $P(n,i,j)$ denote the number of plane trees with n edges, i old leaves and j young leaves, we see that

$$G_n(x_1,x_2) = \sum_{i,j} P(n,i,j)x_1^i x_2^j.$$

By applying the Lagrange inversion formula to the generating function for $G_n(x_1, x_2)$, Chen, Deutsch and Elizalde obtained the following explicit formula for $P(n, i, j)$:

$$P(n,i,j) = \frac{1}{n}\binom{n}{i}\binom{n-i}{j}\binom{n-i-j}{i-1}. \tag{1.11}$$

Moreover, they established a bijection between the set of plane trees with n edges and the set of 2-Motzkin paths of length $n-1$, leading to the following refinement of Coker's formula: For $n \geq 0$,

$$\begin{aligned}&\sum_{i=1}^{\lfloor\frac{n+2}{2}\rfloor}\sum_{j=0}^{n+2-2i}\frac{1}{n+1}\binom{n+1}{i}\binom{n+1-i}{j}\binom{n+1-i-j}{i-1}x_1^i x_2^j\\&=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor} C_k\binom{n}{2k}x_1^{k+1}(1+x_2)^{n-2k}.\end{aligned} \tag{1.12}$$

Using the notation $G_n(x_1, x_2)$ and $M_n(u; v)$, we can state the identity (1.12) as follows:

Theorem 1.5 (Chen-Deutsch-Elizalde)**.** *For $n \geq 0$,*

$$G_{n+1}(x_1,x_2) = M_n(x_1; 1+x_2). \tag{1.13}$$

In fact, this relation can be immediately derived by comparing the generating function (1.10) for $G_n(x_1, x_2)$ with the generating function (1.7) for $M_n(u; v)$.

In this paper, we further refine the Narayana polynomials by considering leaves without any siblings. It should be emphasized that Chen, Deutsch and Elizalde [8] classified a leaf without any siblings as an old leaf. More precisely, we define a leaf without any siblings as a singleton leaf. A leaf with siblings is considered as an elder leaf if it is the leftmost child of its parent, and a young leaf otherwise. An interior vertex is called a young interior vertex if it is not a parent of a singleton leaf or an elder leaf.

Let $\mathrm{sleaf}(T)$ and $\mathrm{eleaf}(T)$ denote the numbers of singleton leaves and elder leaves in a plane tree T, respectively. Let $\mathrm{sint}(T)$ and $\mathrm{eint}(T)$ denote the numbers of parents of singleton leaves and elder leaves in T respectively, and let $\mathrm{yint}(T)$ denote the number of young interior vertices in T. By definition, it is evident that

$$\mathrm{sleaf}(T) = \mathrm{sint}(T), \quad \mathrm{eleaf}(T) = \mathrm{eint}(T)$$

and for $T \in \mathcal{P}_n$,

$$2\mathrm{sleaf}(T) + 2\mathrm{eleaf}(T) + \mathrm{yleaf}(T) + \mathrm{yint}(T) = n + 1.$$

We define the following polynomials:

Definition 1.1 (New refinement of the Narayana polynomials)**.** *For $n \geq 1$,*

$$\begin{aligned} &G_n(x_{11}, x_{12}, x_2; y_{11}, y_{12}, y_2) \\ &= \sum_{T \in \mathcal{P}_n} x_{11}^{\mathrm{sleaf}(T)} x_{12}^{\mathrm{eleaf}(T)} x_2^{\mathrm{yleaf}(T)} y_{11}^{\mathrm{sint}(T)} y_{12}^{\mathrm{eint}(T)} y_2^{\mathrm{yint}(T)} \end{aligned} \tag{1.14}$$

with the convention that $G_0(x_{11}, x_{12}, x_2; y_{11}, y_{12}, y_2) = y_2$.

Fig. 3 gives the list of plane trees in $\mathcal{P}_3$, where the singleton leaves are labeled by x_{11}, the elder leaves are labeled by x_{12}, the young leaves are labeled by x_2, the parents of singleton leaves are labeled by y_{11}, the parents of elder leaves are labeled by y_{12}, and the young interior vertices are labeled by y_2. We have

$$\begin{aligned} &G_3(x_{11}, x_{12}, x_2; y_{11}, y_{12}, y_2) \\ &= x_{11}y_{11}y_2^2 + x_{12}x_2y_{12}y_2 + x_{11}x_2y_{11}y_2 + x_{11}x_{12}y_{11}y_{12} + x_{12}x_2^2y_{12}, \end{aligned}$$

which is a homogeneous polynomial of degree four.

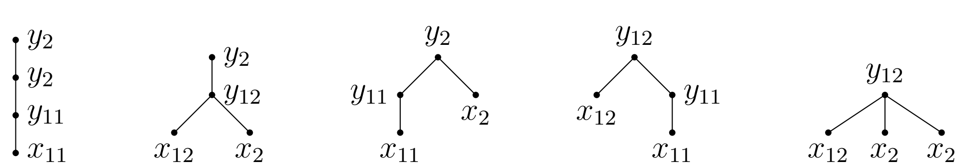


FIGURE 3. 3-edge plane trees.

Setting $x_{11} = x_{12} = x_1$ and $y_{11} = y_{12} = y_2 = 1$ in $G_n(x_{11}, x_{12}, x_2; y_{11}, y_{12}, y_2)$ yields the polynomial $G_n(x_1, x_2)$ as introduced by Chen, Deutsch and Elizalde. Setting $x_{11} = x_{12} = x_2 = x$ and $y_{11} = y_{12} = y_2 = 1$ in $G_n(x_{11}, x_{12}, x_2; y_{11}, y_{12}, y_2)$ recovers the Narayana polynomial $N_n(x)$.

Theorem 1.6. *We have*

$$\begin{aligned}&\sum_{n\geq 0} G_n(x_{11}, x_{12}, x_2; y_{11}, y_{12}, y_2)q^n \\ &= \frac{1 + a_0 q + a_1 q^2 - \sqrt{1 + a_2 q + a_3 q^2 + a_4 q^3 + a_5 q^4}}{2q},\end{aligned} \tag{1.15}$$

where

$$\begin{aligned}
a_0 &= y_2 - x_2,\\
a_1 &= x_{11}y_{11} - x_{12}y_{12},\\
a_2 &= -2(x_2 + y_2),\\
a_3 &= (x_2 + y_2)^2 - 2(x_{11}y_{11} + x_{12}y_{12}),\\
a_4 &= 2(x_{11}y_{11} - x_{12}y_{12})(x_2 - y_2),\\
a_5 &= (x_{11}y_{11} - x_{12}y_{12})^2.
\end{aligned}$$

From (1.15), we see that for $n \geq 2$,

$$G_n(x_{11}, x_{12}, x_2; y_{11}, y_{12}, y_2) = G_n(x_{12}, x_{11}, y_2; y_{12}, y_{11}, x_2).$$

More precisely, we have the following combinatorial assertion:

Proposition 1.2. *Let k, l, i, j be non-negative integers satisfying $2k + 2l + i + j \geq 4$. The number of plane trees with k singleton leaves, l elder leaves, i young leaves and j young interior vertices equals the number of plane trees with l singleton leaves, k elder leaves, j young leaves and i young interior vertices.*

In particular, we have the following consequence.

Corollary 1.2. (a) *When $n \geq 2$, the number of plane trees with n edges, k singleton leaves, l elder leaves equals the number of plane trees with n edges, l singleton leaves, k elder leaves.*

(b) *When $n \geq 2$, the number of plane trees with n edges, i young leaves and j young interior vertices equals the number of plane trees with n edges, j young leaves and i young interior vertices.*

It should be noted that Corollary 1.2 (b) can be viewed as the refinement of the symmetry of the Narayana polynomials (Proposition 1.1).

Setting $x_{11} = x_{12} = x_1$ and $y_{11} = y_{12} = y_2 = 1$ in (1.15), we recover the generating function (1.10) for $G_n(x_1, x_2)$. In fact, we can consider the following four-variable polynomials: For $n \geq 1$,

$$G_n(x_1, x_2; y_1, y_2) = \sum_{T \in \mathcal{P}_n} x_1^{\mathrm{oleaf}(T)} x_2^{\mathrm{yleaf}(T)} y_1^{\mathrm{oint}(T)} y_2^{\mathrm{yint}(T)}, \tag{1.16}$$

where $\mathrm{oint}(T)$ counts the number of old interior vertices, i.e., the vertices that are the parents of old leaves. As a preliminary condition, we set $G_0(x_1, x_2; y_1, y_2) = y_2$. It is evident that

$$\mathrm{oint}(T) = \mathrm{eint}(T) + \mathrm{sint}(T),$$

and so for $n \geq 0$,

$$G_n(x_1, x_1, x_2; y_1, y_1, y_2) = G_n(x_1, x_2; y_1, y_2).$$

Thus, the generating function for $G_n(x_1, x_2; y_1, y_2)$ is derived by setting $x_{11} = x_{12} = x_1$ and $y_{11} = y_{12} = y_1$ in (1.15).

Theorem 1.7. *We have*

$$\sum_{n \geq 0} G_n(x_1, x_2; y_1, y_2) q^n = \frac{1 + (y_2 - x_2)q - \sqrt{1 - 2(x_2 + y_2)q + ((x_2 + y_2)^2 - 4x_1 y_1)q^2}}{2q}.$$

Comparing Theorem 1.7 with Theorem 1.2, we arrive at

Theorem 1.8. *For $n \geq 0$,*

$$G_{n+1}(x_1, x_2; y_1, y_2) = M_n(x_1 y_1; x_2 + y_2). \tag{1.17}$$

Substituting $y_1 = y_2 = 1$ into (1.17) yields Theorem 1.5. Setting $x_1 = x_2 = x$ and $y_1 = y_2 = y$ in (1.17) leads to a result established by Chen and Pan [17, Theorem 1.3 (1.14)] via combinatorial construction.

Setting $x_2 = -y_2$ in (1.17) and using (1.16), we obtain the following combinatorial assertions:

Corollary 1.3. (a) *For $n \geq 1$, the Catalan numbers C_n count the number of plane trees with $2n + 1$ edges and $n + 1$ old leaves.*

(b) *Let $P_{ey}(n, m)$ (resp. $P_{oy}(n, m)$) denote the number of plane trees with n edges, m old leaves and an even (resp. odd) number of young leaves. For $n \geq 1$ and $1 \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor$,*

$$P_{ey}(n, m) - P_{oy}(n, m) = 0.$$

Remark 1. *Corollary 1.3 (a) can be proved by attaching an old leaf to each vertex of plane trees with n edges and Corollary 1.3 (b) can be established combinatorially using the involution of Chen, Shapiro and Yang in [15, Theorem 2.1].*

Observe that

$$\begin{aligned} M_n(x_1y_1; x_2+y_2) &= \sum_{k\geq 0} \binom{n}{2k} C_k (x_1y_1)^{k+1}(x_2+y_2)^{n-2k} \\ &= \sum_{k\geq 0} \binom{n}{2k} C_k (x_1y_1)^{k+1} \sum_{i\geq 0} \binom{n-2k}{i} x_2^{n-2k-i} y_2^i \\ &= \sum_{i\geq 0}\sum_{k\geq 0} \binom{n}{i}\binom{n-i}{2k} C_k x_2^{n-2k-i} y_2^i (x_1y_1)^{k+1} \\ &= \sum_{i=0}^{n} \binom{n}{i} y_2^i M_{n-i}(x_1y_1; x_2). \end{aligned}$$

Hence, by (1.17), we have

Corollary 1.4. *For* $n \geq 0$,

$$G_{n+1}(x_1, x_2; y_1, y_2) = \sum_{i=0}^{n} \binom{n}{i} y_2^i M_{n-i}(x_1y_1; x_2).$$

Setting $x_1 = x_2 = x$ and $y_1 = y_2 = 1$ in Corollary 1.4 results in the following identity established by Lin and Kim [25]:

$$N_{n+1}(x) = \sum_{k=0}^{n} \binom{n}{k} M_k(x; x). \tag{1.18}$$

When $x = 1$, this polynomial reduces to the Euler transformation (1.5) (b) of the Motzkin numbers and the Catalan numbers.

By setting $y_1 = 1$ and $y_2 = 0$ in (1.17), we obtain the following combinatorial interpretation of the Motzkin polynomials due to Donaghey [20]:

$$M_n(x_1; x_2) = G_{n+1}(x_1, x_2; 1, 0). \tag{1.19}$$

More precisely, let $\mathcal{T}_n$ denote the set of plane trees with n edges without young interior vertices. According to (1.19), we have

$$M_n(u; v) = \sum_{T\in\mathcal{T}_{n+1}} u^{\mathrm{oleaf}(T)} v^{\mathrm{yleaf}(T)}. \tag{1.20}$$

This kind of plane trees was called by Donaghey [20] as the *tip-augmented* plane trees. In other words, every interior vertex of a *tip-augmented* plane tree is the parent of an old leaf. For example, only the last two plane trees in Fig. 3 are the tip-augmented plane trees, and so

$$M_2(u; v) = u^2 + uv^2.$$

Furthermore, Fig. 4 shows four tip-augmented plane trees with four edges, where the old leaves are labeled by u and the young leaves are labeled by v. We see that

$$M_3(u; v) = uv^3 + 3u^2v.$$

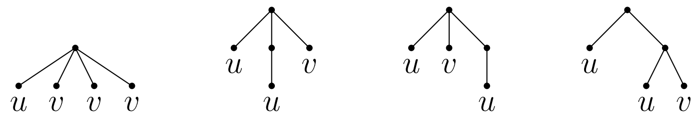


FIGURE 4. 4-edge tip-augmented plane trees.

To achieve further refinement of Coker's formula based on the polynomial $G_n(x_{11}, x_{12}, x_2; y_{11}, y_{12}, y_2)$, we consider a refinement of the Motzkin polynomials. This refinement classifies the old leaves of a tip-augmented plane tree into three categories and the young leaves into two categories. More precisely,

- A leaf without any siblings is considered as a *singleton leaf*.
- A leaf with siblings is considered as an *elder twin leaf* if it is the leftmost child of its parent and the second child of its parent is also a leaf.
- A leaf with siblings is considered as an *elder non-twin leaf* if it is the leftmost child of its parent and the second child of its parent is not a leaf.
- A leaf with siblings is considered as a *second leaf* if it is the second child of its parent.
- A leaf with siblings is considered as a *younger leaf* if it is neither the first child nor the second child of its parent.

Let $\mathrm{etleaf}(T)$, $\mathrm{entleaf}(T)$, $\mathrm{syleaf}(T)$ and $\mathrm{yerleaf}(T)$ denote the numbers of elder twin leaves, elder non-twin leaves, second leaves and younger leaves, respectively.

Definition 1.2 (Refinement of the Motzkin polynomials)**.** *For $n \geq 1$,*

$$M_n(u_1, u_2, u_3; v_1, v_2) = \sum_{T \in \mathcal{T}_{n+1}} u_1^{\mathrm{sleaf}(T)} u_2^{\mathrm{etleaf}(T)} u_3^{\mathrm{entleaf}(T)} v_1^{\mathrm{yerleaf}(T)} v_2^{\mathrm{syleaf}(T)}$$

with $M_0(u_1, u_2, u_3; v_1, v_2) = u_3$ by convention.

Fig. 5 shows further labeling of four tip-augmented plane trees with four edges, where the singleton leaves are labeled by u_1, the elder twin leaves are labeled by u_2, the elder non-twin leaves are labeled by u_3, the second leaves are labeled by v_2 and the younger leaves are labeled by v_1. So

$$M_3(u_1, u_2, u_3; v_1, v_2) = u_2 v_1^2 v_2 + u_1 u_3 v_1 + u_1 u_2 v_2 + u_2 u_3 v_2.$$

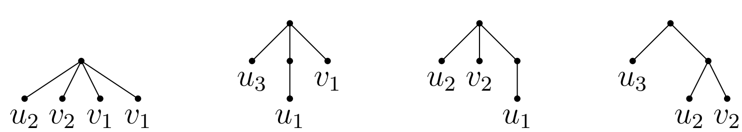


FIGURE 5. 4-edge tip-augmented plane trees.

Theorem 1.9. *We have*

$$\sum_{n\geq 0} M_n(u_1,u_2,u_3;v_1,v_2)q^n$$
$$= \frac{1-v_1q+(u_3-u_1)q^2}{2q^2}$$
$$- \frac{\sqrt{1-2v_1q+(v_1^2-2(u_1+u_3))q^2+2(u_1v_1+u_3v_1-2u_2v_2)q^3+(u_3-u_1)^2q^4}}{2q^2}.$$

Setting $u_1 = u_2 = u_3 = u$ and $v_1 = v_2 = v$ in Theorem 1.9 results in the generating function (1.7) established by Sun. When $u_2 = u_3 = v_1 = v_2 = 1$, we could recover the generating function of the number of (sharp) peaks in Motzkin paths given by Brennan and Mavhungu [3] with the help of the bijection between the set of tip-augmented plane trees and the set of elevated Motzkin paths, see [38].

By applying Theorem 1.9, we find that for $n \geq 2$,

$$M_n(u_1,u_2,u_3;v_1,v_2) = M_n(u_3,u_2,u_1;v_1,v_2),$$

which leads to the following combinatorial assertion:

Proposition 1.3. *Let i,j,k,r,s be non-negative integers satisfying $2(i+j+k)+(r+s) \geq 3$. The number of tip-augmented plane trees with i singleton leaves, j elder twin leaves, k elder non-twin leaves, r younger leaves and s second leaves equals the number of tip-augmented plane trees with k singleton leaves, j elder twin leaves, i elder non-twin leaves, r younger leaves and s second leaves.*

Remark 2. *It would be interesting to give a bijective proof of Proposition 1.3.*

By comparing Theorem 1.9 with Theorem 1.6, we derive the following refinement of Coker's formula.

Theorem 1.10. *For $n \geq 1$,*

$$G_{n+1}(x_{11},x_{12},x_2;y_{11},y_{12},y_2) = M_n(u_1,u_2,u_3;v_1,v_2)$$

with $u_2v_2 = x_{12}y_{12}x_2 + x_{11}y_{11}y_2$, $v_1 = x_2 + y_2$, $u_3 = x_{11}y_{11}$ and $u_1 = x_{12}y_{12}$.

By setting $x_{11} = x_{12} = x_1$, $y_{11} = y_{12} = y_1$, $u_1 = u_2 = u_3 = u$ and $v_1 = v_2 = v$, we retrieve Theorem 1.8.

As stated at the beginning of this paper, the primary objective of this paper is to introduce a grammatical approach to the study of the Narayana polynomials and the Motzkin polynomials. This technique, which employs context-free grammars to explore combinatorial polynomials, was pioneered by Chen [5].

In this paper, we first establish the grammars for the polynomial $G_n(x_{11}, x_{12}, x_2; y_{11}, y_{12}, y_2)$ and the polynomial $M_n(u_1, u_2, u_3; v_1, v_2)$ (see Theorem 3.1 and Theorem 4.1). We then provide grammatical derivations for Theorem 1.6 and Theorem 1.9. As will be demonstrated later, the derivations of the generating functions for $G_n(x_{11}, x_{12}, x_2; y_{11}, y_{12}, y_2)$ and $M_n(u_1, u_2, u_3; v_1, v_2)$ become quite simple once their grammars are established. In fact, relying merely on grammars, we could deduce various generating functions and identities without minding the recurrence relations or differential equations. Additionally, the grammatical approach is also applied to devise bijections [11, 12], prove the γ-positivity of combinatorial polynomials [10, 13, 26, 27] and the stability of multivariable combinatorial polynomials [14, 41]. In this regard, we anticipate further applications of these two grammars.

It is worth noting that the grammar for $G_n(x_{11}, x_{12}, x_2; y_{11}, y_{12}, y_2)$ presented in (3.2) and the grammar for $M_n(u_1, u_2, u_3; v_1, v_2)$ provided in (4.1) can be specialized to yield the grammars for the polynomial $G_n(x_1, x_2; y_1, y_2)$ introduced by Chen, Deutsch and Elizalde, as well as the Motzkin polynomial $M_n(u; v)$ described in Theorem 1.11 and Theorem 1.12, respectively. Concerning the definition of grammars and backgrounds, please see Section 2. The grammar for the Narayana polynomials has been established by Ma, Ma and Yeh [27] and Yang and Zhang [41].

Theorem 1.11. *Let D be the formal derivative with respect to the grammar*

$$G = \{x_2 \to 2tx_1y_1,\ y_2 \to 2tx_1y_1,\ x_1y_1 \to 2tx_1y_1(x_2 + y_2),\ t \to t^2(x_2 + y_2)\}. \tag{1.21}$$

For $n \geq 0$,

$$D^n(y_2) = (n + 1)!t^nG_n(x_1, x_2; y_1, y_2).$$

Theorem 1.12. *Let D be the formal derivative with respect to the grammar*

$$M = \{t \to t^2v,\ u \to 2tuv,\ v \to 4tu\}. \tag{1.22}$$

For $n \geq 1$,

$$D^n\left(\frac{v}{2}\right) = (n + 1)!t^nM_{n-1}(u; v).$$

To conclude the introduction, let us briefly discuss the real-rootedness of the Motzkin polynomials and those defined by Chen, Deutsch and Elizalde [8]. This can be viewed as another application of Coker's formula.

Define

$$M_n(x) := \sum_{T\in\mathcal{T}_{n+1}} x^{\mathrm{oleaf}(T)} \quad \text{and} \quad G_n(x) := \sum_{T\in\mathcal{P}_n} x^{\mathrm{oleaf}(T)}, \tag{1.23}$$

where $\mathcal{T}_n$ denotes the set of tip-augmented plane trees with n edges and $\mathcal{P}_n$ denotes the set of plane trees with n edges. We have the following consequence:

Theorem 1.13. *For $n \geq 1$, the polynomials $G_n(x)$ and $M_n(x)$ have only non-positive real zeros.*

Theorem 1.13 can be derived by applying the fact that the Narayana polynomial $N_n(x)$ has only non-positive real zeros, Coker's formula (1.8), the relation (1.13) and the following observation stated by Petersen [31, Observation 4.2].

Proposition 1.4. [31, Observation 4.2] *If a polynomial $f(x)$ is symmetric, then*

$$f(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_k x^k (1+x)^{n-2k}$$

has only negative (non-positive) real zeros if and only if

$$\gamma(f;x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_k x^k$$

has only negative (non-positive) real zeros.

This paper is organized as follows. Section 2 aims to illustrate the idea of the grammatical approach through a grammatical calculus for the Motzkin polynomials. In Section 3, we provide a grammatical derivation of Theorem 1.6. This involves establishing the grammar for the polynomial $G_n(x_{11}, x_{12}, x_2; y_{11}, y_{12}, y_2)$ (see Theorem 3.1). Section 4 focuses on the grammatical derivation of Theorem 1.9. The grammar for the polynomial $M_n(u_1, u_2, u_3; v_1, v_2)$ is detailed in Theorem 4.1.

2. A GRAMMATICAL CALCULUS FOR THE MOTZKIN POLYNOMIALS

To demonstrate how the grammatical approach works, we first prove Theorem 1.12 by using the grammatical labeling and then derive the generating function for $M_n(u; v)$ (Theorem 1.2) solely based on the grammar in Theorem 1.12.

A context-free grammar G over a set $V = \{x, y, z, \dots\}$ of variables is a set of substitution rules that replace a variable in V by a Laurent polynomial of variables in V. For a context-free grammar G over V, the formal derivative D with respect to G is defined as a linear operator acting on Laurent polynomials with variables in V such that each substitution rule is treated as the common differential rule that satisfies the following relations:

$$D(u+v) = D(u) + D(v), \tag{2.1}$$

$$D(uv) = D(u)v + uD(v). \tag{2.2}$$

Hence, it obeys the Leibniz's rule

$$D^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D^k(u) D^{n-k}(v).$$

For a constant c, we have $D(c) = 0$.

A formal derivative D with respect to G is also associated with an exponential generating function. For a Laurent polynomial w of variables in V, let

$$\mathrm{Gen}(w;q) = \sum_{n\geq 0} D^n(w)\frac{q^n}{n!}. \tag{2.3}$$

Then, by (2.1) and (2.2), we derive that

$$\mathrm{Gen}(u+v;q) = \mathrm{Gen}(u;q) + \mathrm{Gen}(v;q), \tag{2.4}$$

$$\mathrm{Gen}(uv;q) = \mathrm{Gen}(u;q)\mathrm{Gen}(v;q). \tag{2.5}$$

For more information on the grammatical calculus, we refer to Chen [5] and Chen and Fu [9, 10].

Next, we apply the grammatical labeling to prove Theorem 1.12. The notion of a grammatical labeling was introduced by Chen and Fu [9]. Here we need to consider labeled tip-augmented plane trees. Recall that a tip-augmented plane tree is a plane tree without young interior vertices. A labeled plane tree with n edges is a plane tree where each vertex is uniquely labeled with a number from the set $[n+1] = \{1, 2, \ldots, n+1\}$. Let $\mathcal{LT}_n$ denote the set of all labeled tip-augmented plane trees with n edges.

For $T \in \mathcal{LT}_n$, we define the grammatical labeling of T as follows:

- If the vertex **j** of T is an old leaf, then label **j** by u.
- If the vertex **j** of T is a young leaf, then label **j** by v.
- All of edges of T are labeled by t.

The weight of T is defined to be the product of all the labels, that is,

$$\mathrm{wt}(T) = u^{\mathrm{oleaf}(T)} v^{\mathrm{yleaf}(T)} t^{\mathrm{edge}(T)}.$$

For example, Fig. 6 shows the grammatical labeling of a tip-augmented plane tree $T \in \mathcal{LT}_7$ whose weight is $\mathrm{wt}(T) = t^7u^3v^2$. The labels are shown in parentheses. We refer to this labeling scheme of labeled tip-augmented plane trees as the $(u, v; t)$-labeling.

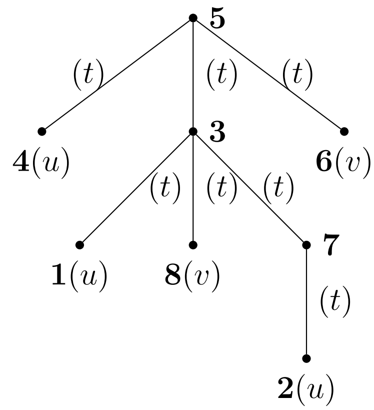


FIGURE 6. A tip-augmented plane tree in $\mathcal{LT}_7$ with the $(u, v; t)$-labeling.

From the definition of the $(u, v; t)$-labeling, we see that the polynomial $M_n(u; v)$ can be interpreted as:

$$(n+2)!t^{n+1}M_n(u; v) = \sum_{T \in \mathcal{LT}_{n+1}} \mathrm{wt}(T). \tag{2.6}$$

Therefore, the proof of Theorem 1.12 is equivalent to the proof of the following assertion: For $n \geq 1$,

$$D^n\left(\frac{v}{2}\right) = \sum_{T \in \mathcal{LT}_n} \mathrm{wt}(T), \tag{2.7}$$

where D is the formal derivative with respect to the grammar defined in (1.22).

Proof of Theorem 1.12. We proceed by induction on n. For $n = 1$, there are two labeled tip-augmented plane trees. The $(u, v; t)$-labelings of those two trees are given in Fig. 7. It is easy to check that the relation (2.7) is valid for $n = 1$.

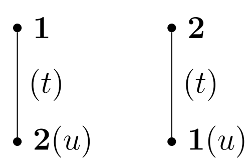


FIGURE 7. Two tip-augmented plane trees in $\mathcal{LT}_1$ with the $(u, v; t)$-labeling.

Assume that this assertion holds for $n-1$, that is, the relation (2.7) is valid for $n-1$. To demonstrate that it also holds for n, it suffices to show, by (2.7), that

$$D\left(\sum_{T \in \mathcal{LT}_{n-1}} \mathrm{wt}(T)\right) = \sum_{T \in \mathcal{LT}_n} \mathrm{wt}(T). \tag{2.8}$$

We will accomplish this by constructing all tip-augmented plane trees $\widetilde{T}$ in $\mathcal{LT}_n$ through the insertion of the vertex $\mathbf{n+1}$ into tip-augmented plane trees in $\mathcal{LT}_{n-1}$, ensuring that the change of weights between these two plane trees adheres to the substitution rules given in the grammar (1.22).

Given a tip-augmented plane tree $T \in \mathcal{LT}_{n-1}$, and let $\widetilde{T}$ be the plane tree with $n+1$ vertices obtained by performing a specific operation at the position $\diamond$ of T. We consider the following three cases:

(1) The position $\diamond$ is an old leaf $\mathbf{j}$ of T labeled by u. Suppose that $\mathbf{i}$ is the parent of $\mathbf{j}$, see Fig. 8. Add the vertex $\mathbf{n+1}$ as illustrated in Fig. 9 to construct the plane tree $\widetilde{T}$. It is easy to check that the change of weights follows the substitution rule $u \to 2tuv$. Moreover, every interior vertex of the plane tree $\widetilde{T}$ has an old leaf, so the plane tree $\widetilde{T}$ is a tip-augmented plane tree.

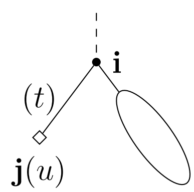
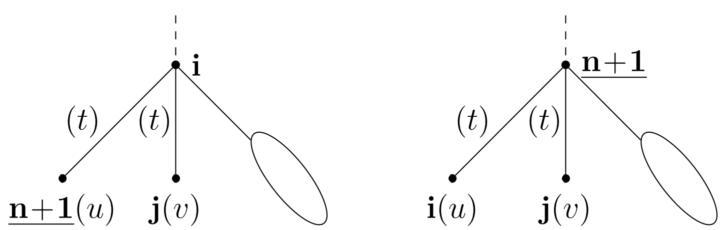


FIGURE 8. The position $\diamond$ is an old leaf of T.



(A): $\mathbf{n+1}$ is an old leaf of $\widetilde{T}$ with at least one sibling, and the sibling immediately following $\mathbf{n+1}$ is a young leaf.

(B): $\mathbf{n+1}$ is an interior vertex of $\widetilde{T}$ with at least two children and its second child is a young leaf.

FIGURE 9. $u \to 2tuv$.

(2) The position $\diamond$ is a young leaf $\mathbf{j}$ of T labeled by v, see Fig. 10. It should be noted that the forest A formed by the subtrees $\mathbf{k_2}, \ldots, \mathbf{k_{t-1}}$ and the forest B formed by the subtrees $\mathbf{k_{t+1}}, \ldots, \mathbf{k_q}$ may be empty. We perform the steps shown in Fig. 11 to create four kinds of tip-augmented plane trees. It is easy to check that the change of weights follows the substitution rule $v \to 4tu$.

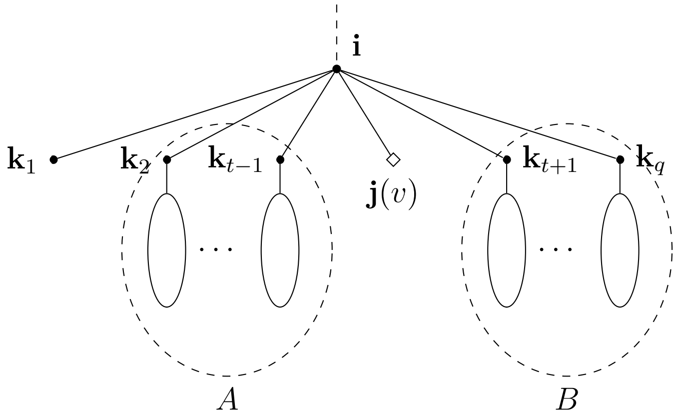
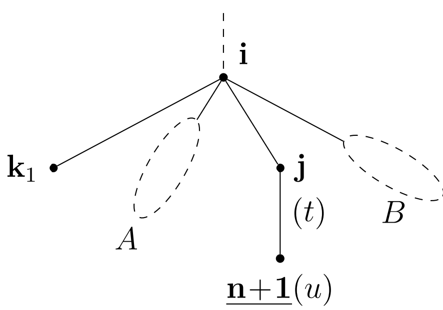
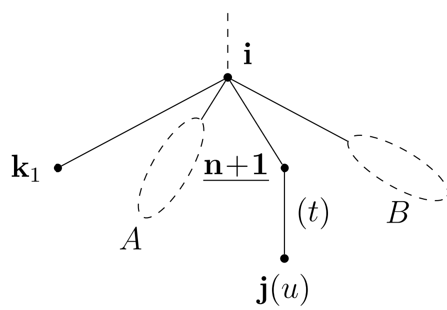


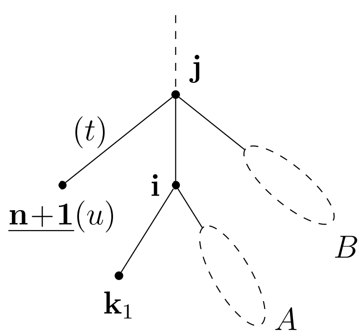
FIGURE 10. The position $\diamond$ is a young leaf of T.



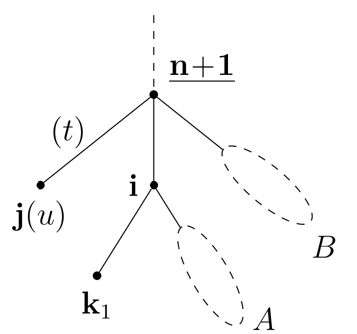
(A): $\mathbf{n+1}$ is an old leaf of $\widetilde{T}$ with no siblings.



(B): $\mathbf{n+1}$ is an interior vertex of $\widetilde{T}$ with only one child.



(C): $\mathbf{n+1}$ is an old leaf of $\widetilde{T}$ with at least one sibling, and the sibling immediately following $\mathbf{n+1}$ is not a leaf.



(D): $\mathbf{n+1}$ is an interior vertex of $\widetilde{T}$ with at least two children and its second child is not a leaf.

FIGURE 11. $v \to 4tu$.

(3) The position $\diamond$ is an edge $(\mathbf{i}, \mathbf{j})$ in T labeled by t, as shown in Fig. 12. Insert the edge $(\mathbf{i}, \mathbf{n+1})$ right immediately after $(\mathbf{i}, \mathbf{j})$ as illustrated in Fig. 13 (A) to get the tip-augmented plane tree $\widetilde{T}$. The change of weights is in accordance with $t \to t^2 v$.

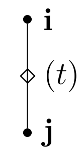
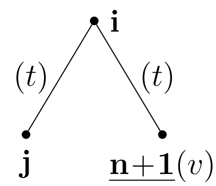


FIGURE 12. The position $\diamond$ is an edge.



(A): $\mathbf{n+1}$ is a young leaf of $\widetilde{T}$.

FIGURE 13. $t \to t^2 v$.

From the above construction, it is clear that $\widetilde{T} \in \mathcal{LT}_n$. Moreover,

- $\mathbf{n+1}$ is an old leaf of $\widetilde{T}$ in Fig. 9 (A), Fig. 11 (A) and Fig. 11 (C) .
- $\mathbf{n+1}$ is a young leaf of $\widetilde{T}$ in Fig. 13 (A).
- $\mathbf{n+1}$ is an interior vertex of $\widetilde{T}$ in Fig. 9 (B), Fig. 11 (B) and Fig. 11 (D).

Given this observation, we see that the construction is reversible depending on the position of the vertex $\mathbf{n+1}$ in $\widetilde{T}$. Consequently, we could generate all tip-augmented plane trees in $\mathcal{LT}_n$ from those in $\mathcal{LT}_{n-1}$ using the above construction. Moreover, the change of weights between tip-augmented plane trees in $\mathcal{LT}_n$ and those in $\mathcal{LT}_{n-1}$ is consistent with the substitution rule given in the grammar (1.22). Thus we show that (2.7) is valid and so the assertion is also valid for n. This completes the proof. $\square$

Equipped with Theorem 1.12, we are now in a position to provide a grammatical derivation of the generating function for $M_n(u; v)$ as stated in Theorem 1.2.

The grammatical derivation for Theorem 1.2. For the formal derivative D with respect to the grammar M in (1.22), we observe that

$$D(t^{-1}) = -v \quad \text{and} \quad D^2(t^{-1}) = -D(v).$$

Combining this with Theorem 1.12, we deduce that for $n \geq 2$,

$$D^n(t^{-1}) = -2n! t^{n-1} M_{n-2}(u; v).$$

Consequently,

$$\begin{aligned}
\mathrm{Gen}(t^{-1};q) &= \sum_{n\geq 0} D^n(t^{-1})\frac{q^n}{n!} \\
&= t^{-1} - vq - \sum_{n\geq 2} D^{n-1}(v)\,\frac{q^n}{n!} \\
&= t^{-1} - vq - 2\sum_{n\geq 2} n!t^{n-1}M_{n-2}(u;v)\frac{q^n}{n!} \\
&= t^{-1} - vq - 2tq^2\sum_{n\geq 0} M_n(u;v)t^nq^n. \qquad (2.9)
\end{aligned}$$

Since

$$\mathrm{Gen}(t^{-2};q) = \mathrm{Gen}^2(t^{-1};q), \qquad (2.10)$$

it suffices to compute $\mathrm{Gen}(t^{-2};q)$. Observe that

$$D(t^{-2}) = -2t^{-1}v, \quad D^2(t^{-2}) = 2v^2 - 8u, \quad \text{and} \quad D^3(t^{-2}) = 0.$$

It follows that

$$\mathrm{Gen}(t^{-2};q) = \sum_{n\geq 0} D^n(t^{-2})\frac{q^n}{n!} = t^{-2} - 2t^{-1}vq + (v^2-4u)q^2. \qquad (2.11)$$

By setting $q=0$ in (2.10), we infer from (2.9) and (2.11) that

$$\mathrm{Gen}(t^{-1};q) = \sqrt{t^{-2} - 2t^{-1}vq + (v^2-4u)q^2}. \qquad (2.12)$$

Substituting (2.12) into (2.9) and setting $t=1$, we arrive at the generating function for $M_n(u;v)$ as stated in Theorem 1.2. □

3. Proof of Theorem 1.6

The main objective of this section is to give a proof of Theorem 1.6 by using the grammatical calculus. To this end, we need to establish the grammar for the polynomial $G_n(x_{11}, x_{12}, x_2; y_{11}, y_{12}, y_2)$. It turns out that we should classify the edges of a plane tree. An edge is called a young edge if it is not an edge of an old leaf (including a singleton leaf and an elder leaf). Let $\mathrm{yedge}(T)$ denote the number of young edges of a plane tree T.

For $n\geq 1$, we define the polynomial $\widetilde{G}_n(a,b,c,d;t)$ as follows:

$$\widetilde{G}_n(a,b,c,d;t) = \sum_{T\in\mathcal{P}_n} a^{\mathrm{sleaf}(T)}b^{\mathrm{eleaf}(T)}c^{\mathrm{yleaf}(T)}d^{\mathrm{yint}(T)}t^{\mathrm{yedge}(T)} \qquad (3.1)$$

with the convention that $\widetilde{G}_0(a,b,c,d;t) = d$.

By definition, it is evident that for $n\geq 1$,

$$\widetilde{G}_n(x_{11}y_{11}t, x_{12}y_{12}t, x_2, y_2; t) = t^nG_n(x_{11}, x_{12}, x_2; y_{11}, y_{12}, y_2).$$

In this section, we first show that the following grammar

$$G = \{a \to 3t(ad+bc),\ b \to 3t(ad+bc),\ c \to 2a,\ d \to 2b,\ t \to t^2(c+d)\} \tag{3.2}$$

can be used to generate the polynomial $\widetilde{G}_n(a,b,c,d;t)$. More precisely,

Theorem 3.1. *Let D be the formal derivative with respect to the grammar defined in (3.2). For $n \geq 1$,*

$$D^n(c) = (n+1)!\widetilde{G}_n(a,b,c,d;t).$$

The first few values of $D^n(c)$ are given below:

$$\begin{aligned}
D(c) &= 2a,\\
D^2(c) &= 6(tad+tbc),\\
D^3(c) &= 24(tab+t^2acd+t^2ad^2+t^2bc^2+t^2bcd),\\
D^4(c) &= 120(t^2a^2d+2t^2abd+t^3ad^3+t^3ac^2d+2t^3acd^2+2t^2abc\\
&\qquad +t^2b^2c+t^3bc^3+2t^3bc^2d+t^3bcd^2).
\end{aligned}$$

By setting $a=b=x_1y_1t$, $c=x_2$, $d=y_2$ in the grammar G given in (3.2), we obtain the grammar G presented in (1.21) for the polynomial $G_n(x_1,x_2;y_1,y_2)$. Thus, we derive Theorem 1.11 from Theorem 3.1.

With Theorem 3.1 in hand, we obtain the following generating function for $\widetilde{G}_n(a,b,c,d;t)$. By setting $t=1$, $a=x_{11}y_{11}, b=x_{12}y_{12}$, $c=x_2$ and $d=y_2$, it can be specialized to obtain the generating function for $G_n(x_{11},x_{12},x_2;y_{11},y_{12},y_2)$ as stated in Theorem 1.6.

Theorem 3.2. *We have*

$$\begin{aligned}
\sum_{n\geq 0}\widetilde{G}_n(a,b,c,d;t)q^n &= \frac{t^{-1}+(d-c)q+(a-b)q^2}{2q}\\
&- \frac{\sqrt{t^{-2}-2t^{-1}(c+d)q+((c+d)^2-2t^{-1}(a+b))q^2+2(a-b)(c-d)q^3+(a-b)^2q^4}}{2q}.
\end{aligned}$$

3.1. **A grammatical labeling of $\widetilde{G}_n(a,b,c,d;t)$.** In this subsection, we aim to show Theorem 3.1 by using the grammatical labeling. Similarly, we need to consider labeled plane trees. Let $\mathcal{LP}_n$ denote the set of all labeled plane trees with n edges.

The grammatical labeling of a plane tree $T \in \mathcal{LP}_n$ ($n \geq 2$) is defined as follows:

- If the vertex **j** is a singleton leaf, then label **j** by a.
- If the vertex **j** is an elder leaf, then label **j** by b.
- If the vertex **j** is a young leaf, then label **j** by c.
- If the vertex **j** is a young interior vertex, then label **j** by d.

- If the edge $(\mathbf{i}, \mathbf{j})$ is a young edge, then label it by t.

This labeling scheme of labeled plane trees is called the $(a, b, c, d; t)$-labeling.

For example, Fig. 14 gives the grammatical labeling of a plane tree T in $\mathcal{LP}_6$. The labels of T are shown in parentheses.

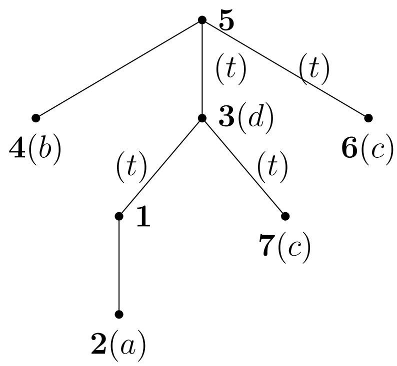


FIGURE 14. A plane tree in $\mathcal{LP}_6$ with the $(a, b, c, d; t)$-labeling.

The weight of T is defined to be the product of all the labels, that is,

$$\mathrm{wt}(T) = a^{\mathrm{sleaf}(T)} b^{\mathrm{eleaf}(T)} c^{\mathrm{yleaf}(T)} d^{\mathrm{yint}(T)} t^{\mathrm{yedge}(T)}.$$

In this case, the weight of T is $\mathrm{wt}(T) = t^4 abc^2 d$.

From the definition of the $(a, b, c, d; t)$-labeling, we see that the polynomial $\widetilde{G}_n(a, b, c, d; t)$ given in (3.1) can be interpreted as:

$$(n+1)!\widetilde{G}_n(a, b, c, d; t) = \sum_{T \in \mathcal{LP}_n} \mathrm{wt}(T) \tag{3.3}$$

and so the proof of Theorem 3.1 is equivalent to the proof of the following assertion: For $n \geq 1$,

$$D^n(c) = \sum_{T \in \mathcal{LP}_n} \mathrm{wt}(T), \tag{3.4}$$

where D is the formal derivative with respect to the grammar defined in (3.2).

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. We proceed by induction on n. By convention, we see that the assertion is trivial for $n = 1$. When $n = 2$, there are two plane trees with two edges and so there are twelve labeled plane trees in $\mathcal{LP}_2$. Fig. 15 gives the grammatical labeling of labeled plane trees in $\mathcal{LP}_2$ with $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ representing their vertices.

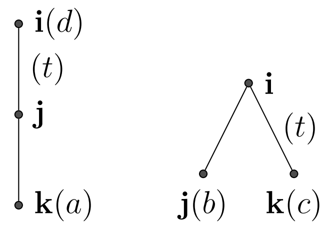


FIGURE 15. Two plane trees in $\mathcal{LP}_2$ with the $(a, b, c, d; t)$-labeling.

It is easy to check that

$$\sum_{T \in \mathcal{LP}_2} \mathrm{wt}(T) = 6tad + 6tbc = D^2(c).$$

Hence this assertion is valid for $n = 2$. Assume that it holds for $n-1$, to show that this assertion is also valid for n, by (3.4), it suffices to show that

$$D\left(\sum_{T \in \mathcal{LP}_{n-1}} \mathrm{wt}(T)\right) = \sum_{T \in \mathcal{LP}_n} \mathrm{wt}(T). \tag{3.5}$$

To this end, we will construct a labeled plane tree $\widetilde{T}$ by inserting the vertex $\mathbf{n+1}$ into a labeled plane tree T so that the change of weights between these two plane trees adheres to the substitution rules given in the grammar (3.2). Given a plane tree $T \in \mathcal{LP}_{n-1}$, and let $\widetilde{T}$ be the plane tree with $n+1$ vertices obtained by performing a specific operation at the position $\diamond$ of T. We consider the following five cases:

(1) The position $\diamond$ is a singleton leaf $\mathbf{j}$ of T labeled by a. Suppose that $\mathbf{i}$ is the parent of $\mathbf{j}$, see Fig. 16. We have two ways to create the plane tree $\widetilde{T}$.

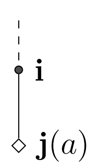
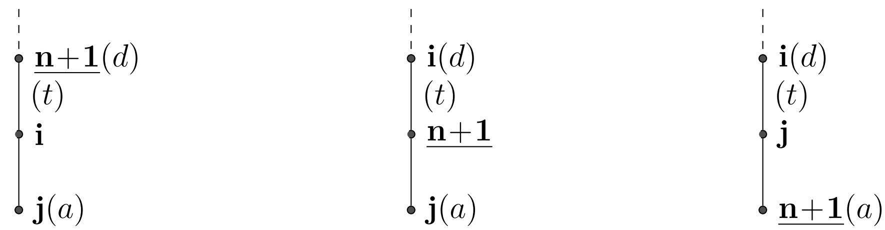


FIGURE 16. The position $\diamond$ is a singleton leaf of T.

(1-i) Insert the vertex $\mathbf{n+1}$ as illustrated in Fig. 17 to create the plane tree $\widetilde{T}$. It is easy to check that the change of weights follows the substitution rule $a \to 3tad$.



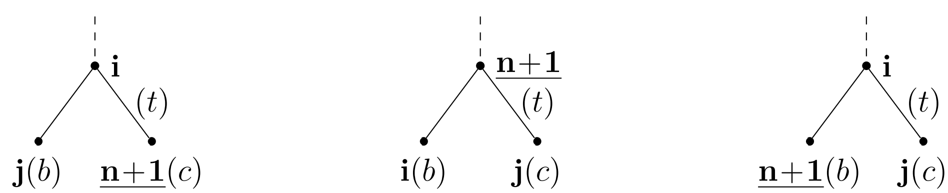
(A): $\mathbf{n+1}$ is a young interior vertex of $\widetilde{T}$ that has only one child, and its child also has only one child.

(B): $\mathbf{n+1}$ is the parent of a singleton leaf in $\widetilde{T}$ and $\mathbf{n+1}$ is the only child of its parent.

(C): $\mathbf{n+1}$ is a singleton leaf of $\widetilde{T}$ and its parent has no siblings.

FIGURE 17. $a \to 3tad$.

(1-ii) Insert the vertex $\mathbf{n+1}$ according to the illustration in Fig. 18 to form the plane tree $\widetilde{T}$. A simple check reveals that the weights change according to the substitution rule $a \to 3tbc$.



(A): $\mathbf{n+1}$ is a young leaf of $\widetilde{T}$ with only one sibling and its sibling is an elder leaf.

(B): $\mathbf{n+1}$ is the parent of an elder leaf of $\widetilde{T}$, which has only two children, and the other child is a young leaf.

(C): $\mathbf{n+1}$ is an elder leaf of $\widetilde{T}$ with only one sibling and its sibling is a young leaf.

FIGURE 18. $a \to 3tbc$.

(2) The position $\diamond$ is an elder leaf $\mathbf{j}$ of T labeled by b, see Fig. 19. It follows the same actions as in Case (1) (see Fig. 20 and Fig. 21). It is straightforward to check that the resulting change in weights adheres to the substitution rule $b \to 3t(ad + bc)$.

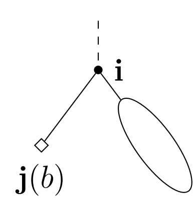
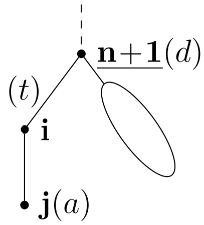
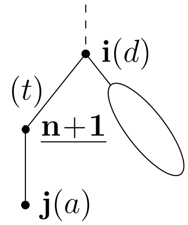


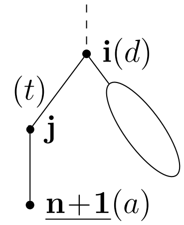
FIGURE 19. The position $\diamond$ is an elder leaf of T.



(A): $\mathbf{n+1}$ is a young interior vertex of $\widetilde{T}$, which has at least two children, and the first child is the parent of a singleton leaf.

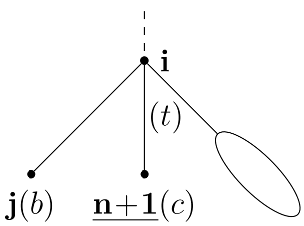


(B): $\mathbf{n+1}$ is the parent of a singleton leaf in $\widetilde{T}$ and $\mathbf{n+1}$ is the first child of its parent.

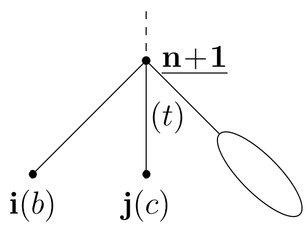


(C): $\mathbf{n+1}$ is a singleton leaf of $\widetilde{T}$, whose parent is the leftmost interior vertex with at least one sibling.

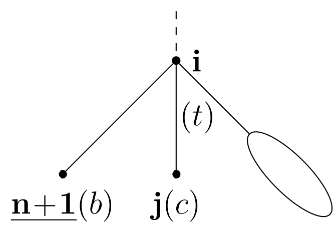
FIGURE 20. $b \to 3tad$.



(A): $\mathbf{n+1}$ is a young leaf of $\widetilde{T}$ with at least two siblings and the sibling immediately preceding $\mathbf{n+1}$ is an elder leaf.



(B): $\mathbf{n+1}$ is the parent of an elder leaf in $\widetilde{T}$, which has at least three children and the second child is a young leaf.



(C): $\mathbf{n+1}$ is an elder leaf of $\widetilde{T}$ with at least two siblings, and the sibling immediately following $\mathbf{n+1}$ is a young leaf.

FIGURE 21. $b \to 3tbc$.

(3) The position $\diamond$ is a young leaf $\mathbf{j}$ of T labeled by c, as depicted in Fig. 22. We perform the steps shown in Fig. 23 to create the plane tree $\widetilde{T}$. It is easy to check that the change in weights follows the substitution rule $c \to 2a$.

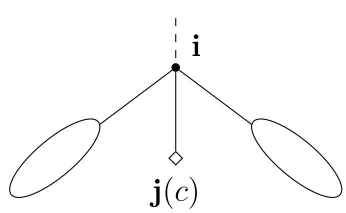
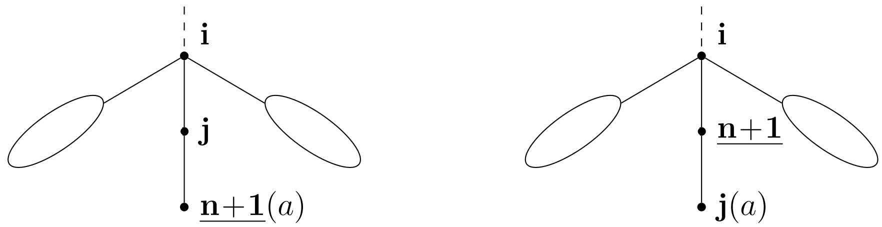


FIGURE 22. The position $\diamond$ is a young leaf of T.



(A): $\mathbf{n+1}$ is a singleton leaf of $\widetilde{T}$ whose parent has at least one elder sibling.

(B): $\mathbf{n+1}$ is the parent of a singleton leaf in $\widetilde{T}$ and $\mathbf{n+1}$ is not the first child of its parent.

FIGURE 23. $c \to 2a$.

(4) The position $\diamond$ is depicted in Fig. 24 as a young interior vertex $\mathbf{i}$ of T labeled by d. We execute the steps shown in Fig. 25 to generate the plane tree $\widetilde{T}$. We see that the change in weights complies with the substitution rule $d \to 2b$.

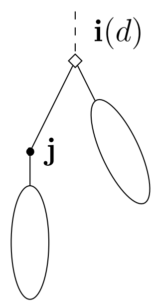
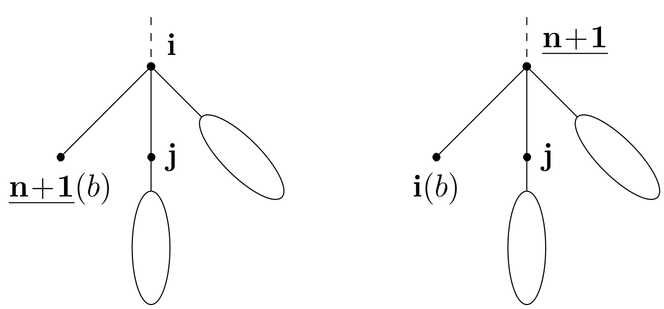


FIGURE 24. The position $\diamond$ is a young interior vertex of T.



(A): $\mathbf{n+1}$ is an elder leaf of $\widetilde{T}$ with at least one sibling and the sibling immediately following $\mathbf{n+1}$ is not a leaf.

(B): $\mathbf{n+1}$ is the parent of an elder leaf in $\widetilde{T}$, whose second child is not a leaf.

FIGURE 25. $d \to 2b$.

(5) The position $\diamond$ is a young edge $(\mathbf{i}, \mathbf{j})$ of T labeled by t as depicted in Fig. 26. We construct the plane tree $\widetilde{T}$ by following the steps shown in Fig. 27. More precisely, suppose that $\mathbf{i}$ has q children, which are $\mathbf{k}_1, \ldots, \mathbf{k}_{t-1}, \mathbf{j}, \mathbf{k}_{t+1}, \ldots, \mathbf{k}_q$. We either add the edge $(\mathbf{i}, \mathbf{n+1})$ immediately after $(\mathbf{i}, \mathbf{j})$ or make $\mathbf{n+1}$ the parent of $\mathbf{i}$. In the latter case, let $\mathbf{i}, \mathbf{k}_{t+1}, \ldots, \mathbf{k}_q$ be the successors of the vertex $\mathbf{n+1}$ (if exists) and let $\mathbf{k}_1, \ldots, \mathbf{k}_{t-1}$ and $\mathbf{j}$ be the children of $\mathbf{i}$ (if exists). It can be checked that the change in weights conforms to the substitution $t \to t^2(c+d)$.

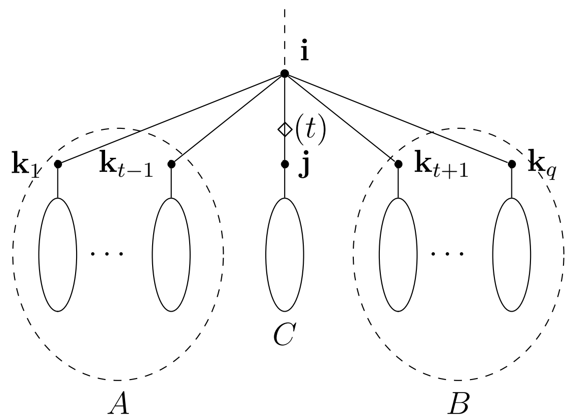
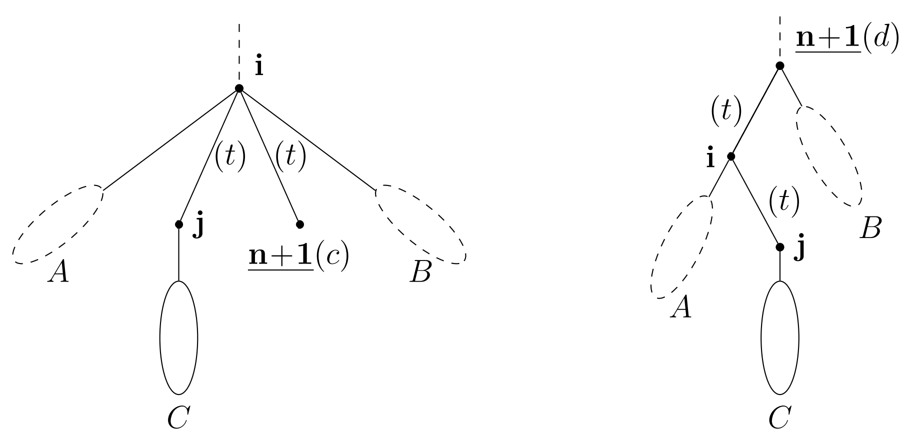


FIGURE 26. The position $\diamond$ is a young edge of T.



(A): $\mathbf{n+1}$ is a young leaf of $\widetilde{T}$ with at least one sibling and the sibling immediately preceding $\mathbf{n+1}$ is not an elder leaf.

(B): $\mathbf{n+1}$ is a young interior vertex of $\widetilde{T}$, whose first child is not the parent of a singleton leaf.

FIGURE 27. $t \to t^2(c+d)$.

From the above construction, we find that

- $\mathbf{n+1}$ is a singleton leaf of $\widetilde{T}$ in Fig. 17 (C), Fig. 20 (C) and Fig. 23 (A) .

- $\mathbf{n+1}$ is an elder leaf of $\widetilde{T}$ in Fig. 18 (C), Fig. 21 (C) and Fig. 25 (A).
- $\mathbf{n+1}$ is a young leaf of $\widetilde{T}$ in Fig. 18 (A), Fig. 21 (A) and Fig. 27 (A).
- $\mathbf{n+1}$ is the parent of a singleton leaf of $\widetilde{T}$ in Fig. 17 (B), Fig. 20 (B) and Fig. 23 (B).
- $\mathbf{n+1}$ is the parent of an elder leaf of $\widetilde{T}$ in Fig. 18 (B), Fig. 21 (B) and Fig. 25 (B).
- $\mathbf{n+1}$ is a young interior vertex of $\widetilde{T}$ in Fig. 17 (A), Fig. 20 (A) and Fig. 27 (B).

Given these observations, it is clear that this construction is reversible according to the position of the vertex $\mathbf{n+1}$ in $\widetilde{T}$. Therefore, we could generate all labeled plane trees in $\mathcal{LP}_n$ from labeled plane trees in $\mathcal{LP}_{n-1}$. Moreover, the change of weights between plane trees in $\mathcal{LP}_n$ and plane trees in $\mathcal{LP}_{n-1}$ in this construction is consistent with the substitution rule given in the grammar (3.2). Thus (3.5) is valid and consequently, the assertion is valid for n as well. This completes the proof. $\square$

3.2. **The grammatical derivation for Theorem 3.2.** For the formal derivative D with respect to the grammar G in (3.2), we find that

$$D(t^{-1}) = -(c+d),$$

$$D^2(t^{-1}) = -2(a+b),$$

$$D^3(t^{-1}) = -2D(2a) = -2D^2(c).$$

Consequently, invoking Theorem 3.1, we derive that for $n \geq 3$,

$$D^n(t^{-1}) = -2n!\widetilde{G}_{n-1}(a,b,c,d;t).$$

Under the assumption $\widetilde{G}_0(a,b,c,d;t) = d$ and based on $\widetilde{G}_1(a,b,c,d;t) = a$, we deduce that

$$\begin{aligned}
\mathrm{Gen}(t^{-1};q) &= \sum_{n\geq 0} D^n(t^{-1})\frac{q^n}{n!} \\
&= t^{-1} - (c+d)q - (a+b)q^2 + \sum_{n\geq 3} D^n(t^{-1})\frac{q^n}{n!} \\
&= t^{-1} - (c+d)q - (a+b)q^2 - 2\sum_{n\geq 3} n!\widetilde{G}_{n-1}(a,b,c,d;t)\frac{q^n}{n!} \\
&= t^{-1} - (c+d)q - (a+b)q^2 - 2q\sum_{n\geq 2} \widetilde{G}_n(a,b,c,d;t)q^n \\
&= t^{-1} - (c+d)q - (a+b)q^2 - 2q\sum_{n\geq 0} \widetilde{G}_n(a,b,c,d;t)q^n + 2dq + 2aq^2 \\
&= t^{-1} + (d-c)q + (a-b)q^2 - 2q\sum_{n\geq 0} \widetilde{G}_n(a,b,c,d;t)q^n. \qquad (3.6)
\end{aligned}$$

In order to derive the generating function for $\widetilde{G}_n(a,b,c,d;t)$, it suffices to compute the generating function $\mathrm{Gen}(t^{-1};q)$. In view of the following relation

$$\mathrm{Gen}(t^{-2};q) = \mathrm{Gen}^2(t^{-1};q), \tag{3.7}$$

it is enough to compute $\mathrm{Gen}(t^{-2};q)$.

Notice that

$$\begin{aligned}
D(t^{-2}) &= -2t^{-1}(c+d),\\
D^2(t^{-2}) &= -4t^{-1}(a+b) + 2(c+d)^2,\\
D^3(t^{-2}) &= 12(a-b)(c-d),\\
D^4(t^{-2}) &= 24(a-b)^2,\\
D^5(t^{-2}) &= 0.
\end{aligned}$$

Hence, we acquire

$$\begin{aligned}
\mathrm{Gen}(t^{-2};q) &= \sum_{n\geq 0} D^n(t^{-2})\frac{q^n}{n!}\\
&= t^{-2} - 2t^{-1}(c+d)q + ((c+d)^2 - 2t^{-1}(a+b))q^2\\
&\quad + 2(a-b)(c-d)q^3 + (a-b)^2q^4.
\end{aligned} \tag{3.8}$$

By setting $q=0$ in (3.7), we derive from (3.6) and (3.8) that

$$\mathrm{Gen}(t^{-1};q) = \sqrt{\mathrm{Gen}(t^{-2};q)}, \tag{3.9}$$

which completes the proof with the aid of (3.6), (3.8) and (3.9). □

4. Proof of Theorem 1.9

To prove Theorem 1.9 by using the grammatical approach, it is essential to establish the grammar for the polynomial $M_n(u_1,u_2,u_3;v_1,v_2)$. Additionally, the concept of the young edge introduced in Section 3 proves to be necessary. Recall that $\mathrm{yedge}(T)$ denotes the number of young edges of a plane tree, i.e., the edges that are not connected to an old leaf. For $n\geq 1$, we define $M_n(u_1,u_2,u_3;v_1,v_2;t)$ as follows:

$$M_n(u_1,u_2,u_3;v_1,v_2;t) = \sum_{T\in\mathcal{T}_{n+1}} u_1^{\mathrm{sleaf}(T)}u_2^{\mathrm{etleaf}(T)}u_3^{\mathrm{entleaf}(T)}v_1^{\mathrm{yerleaf}(T)}v_2^{\mathrm{syleaf}(T)}t^{\mathrm{yedge}(T)}$$

with the convention that $M_0(u_1,u_2,u_3;v_1,v_2;t) = u_3$.

We first show that the following grammar

$$\begin{aligned}
M = \{&u_1\to 3tu_2v_2,\ u_2\to 3tu_2v_1,\ u_3\to 3tu_2v_2,\\
&v_1\to 2(u_1+u_3),\ v_2\to 4u_1u_2^{-1}u_3,\ t\to t^2v_1\}
\end{aligned} \tag{4.1}$$

can be used to generate the polynomial $M_n(u_1,u_2,u_3;v_1,v_2;t)$. To wit,

Theorem 4.1. *Let D be the formal derivative associated with the grammar (4.1). For $n\geq 0$,*

$$D^n(2u_3)=(n+2)!M_n(u_1,u_2,u_3;v_1,v_2;t). \tag{4.2}$$

The first few values of $D^n(2u_3)$ are given below:

$$\begin{aligned}
D(2u_3) &= 6tu_2v_2,\\
D^2(2u_3) &= 24(t^2u_2v_1v_2+tu_1u_3),\\
D^3(2u_3) &= 120(t^3u_2v_1^2v_2+t^2u_1u_2v_2+t^2u_1u_3v_1+t^2u_2u_3v_2),\\
D^4(2u_3) &= 720(t^4u_2v_1^3v_2+t^3u_2^2v_2^2+2t^3u_2u_3v_1v_2+2t^3u_1u_2v_1v_2\\
&\qquad +t^3u_1u_3v_1^2+t^2u_1u_3^2+t^2u_1^2u_3).
\end{aligned}$$

Note that the grammar M given in (1.22) for the Motzkin polynomial $M_n(u;v)$ can be derived by setting $v_1=v_2=v$ and $u_1=u_2=u_3=ut$ in Theorem 4.1.

Armed with Theorem 4.1, we can provide a grammatical derivation of the generating function for $M_n(u_1,u_2,u_3;v_1,v_2;t)$. The special case $t=1$ then yields the generating function for $M_n(u_1,u_2,u_3;v_1,v_2)$ as stated in Theorem 1.9.

Theorem 4.2. *We have*

$$\begin{aligned}
&\sum_{n\geq 0} M_n(u_1,u_2,u_3;v_1,v_2;t)q^n\\
&=\frac{t^{-1}-v_1q+(u_3-u_1)q^2}{2q^2}\\
&\quad-\frac{\sqrt{t^{-2}-2t^{-1}v_1q+(v_1^2-2t^{-1}(u_1+u_3))q^2+(2u_1v_1+2u_3v_1-4u_2v_2)q^3+(u_1-u_3)^2q^4}}{2q^2}.
\end{aligned}$$

4.1. **A grammatical labeling of $M_n(u_1,u_2,u_3;v_1,v_2;t)$.** To prove Theorem 4.1, we need to refine the grammatical labeling of labeled tip-augmented plane trees for the Motzkin polynomial $M_n(u;v)$ introduced in Section 2. More precisely, the refined grammatical labeling of a labeled tip-augmented plane tree $T\in\mathcal{LT}_n$ is defined as follows:

- If the vertex **j** is a singleton leaf, then label **j** by u_1.
- If the vertex **j** is an elder twin leaf, then label **j** by u_2.
- If the vertex **j** is an elder non-twin leaf, then label **j** by u_3.
- If the vertex **j** is a younger leaf, then label **j** by v_1.
- If the vertex **j** is a second leaf, then label **j** by v_2.
- The young edge is labeled by t.

The weight of T is defined to be the product of all the labels, that is,

$$\mathrm{wt}(T) = u_1^{\text{sleaf}(T)} u_2^{\text{etleaf}(T)} u_3^{\text{entleaf}(T)} v_1^{\text{yerleaf}(T)} v_2^{\text{syleaf}(T)} t^{\text{yedge}(T)}.$$

For example, Fig. 28 shows refined grammatical labeling of a tip-augmented plane tree $T \in \mathcal{LT}_7$ whose weight is $\mathrm{wt}(T) = t^4 u_1 u_2 u_3 v_1 v_2$. We refer to this refined labeling of tip-augmented plane trees as the $(u_1, u_2, u_3; v_1, v_2; t)$-labeling.

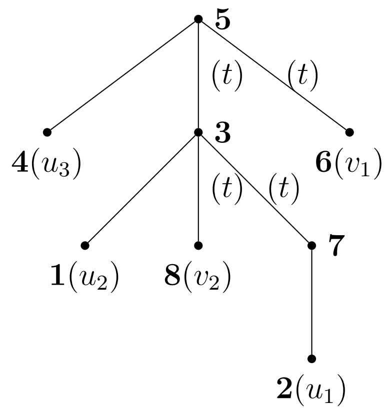


FIGURE 28. A tip-augmented plane tree with the $(u_1, u_2, u_3; v_1, v_2; t)$-labeling.

From the definition of the $(u_1, u_2, u_3; v_1, v_2; t)$-labeling, we see that $M_n(u_1, u_2, u_3; v_1, v_2; t)$ can be interpreted as:

$$(n+2)! M_n(u_1, u_2, u_3; v_1, v_2; t) = \sum_{T \in \mathcal{LT}_{n+1}} \mathrm{wt}(T). \tag{4.3}$$

Proof of Theorem 4.1. We proceed by induction on n. By convention, we see that the assertion is trivial for $n = 0$. For $n = 1$, there are six labeled tip-augmented plane trees T in $\mathcal{LT}_2$. For every plane tree T in $\mathcal{LT}_2$ with $\mathbf{i}, \mathbf{j}, \mathbf{k}$ being its vertices, the $(u_1, u_2, u_3; v_1, v_2; t)$-labeling of T is showed in Fig. 29, and its weight is $\mathrm{wt}(T) = u_2 v_2 t$. As a result, it can be checked that (4.2) is valid for $n = 1$. Assume that this assertion holds for $n - 2$. To show that it also holds for $n - 1$, it suffices to show that

$$D\left(\sum_{T \in \mathcal{LT}_{n-1}} \mathrm{wt}(T)\right) = \sum_{T \in \mathcal{LT}_n} \mathrm{wt}(T), \tag{4.4}$$

where D is the formal derivative with respect to the grammar defined in (4.1). To accomplish this, we will construct tip-augmented plane trees in $\mathcal{LT}_n$ by adding the vertex $\mathbf{n+1}$ to tip-augmented plane trees in $\mathcal{LT}_{n-1}$, ensuring that the change in weights between these two plane trees adheres to the substitution rule given in the grammar (4.1).

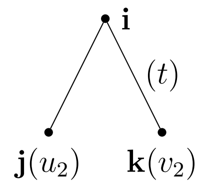


FIGURE 29. A tip-augmented plane tree $T \in \mathcal{LT}_2$ with the $(u_1, u_2, u_3; v_1, v_2; t)$-labeling.

Given a plane tree $T \in \mathcal{LT}_{n-1}$, and let $\widetilde{T}$ be the plane tree with $n+1$ vertices obtained by performing a specific operation at the position $\diamond$ of T. We consider the following six cases:

(1) The position $\diamond$ is a singleton leaf $\mathbf{j}$ of T labeled by u_1. Suppose that $\mathbf{i}$ is the parent of $\mathbf{j}$, see Fig. 30. Insert the vertex $\mathbf{n+1}$ as illustrated in Fig. 31. It is easy to check that the change of weights follows the substitution rule $u_1 \to 3tu_2v_2$.

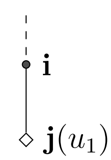
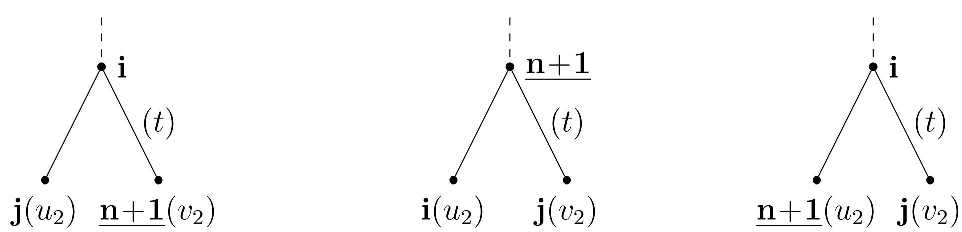


FIGURE 30. The position $\diamond$ is a singleton leaf of T.



(A): $\mathbf{n+1}$ is a second leaf of $\widetilde{T}$ and the parent of $\mathbf{n+1}$ has only two children.

(B): $\mathbf{n+1}$ is the parent of an elder twin leaf in $\widetilde{T}$ and $\mathbf{n+1}$ has only two children.

(C): $\mathbf{n+1}$ is an elder twin leaf of $\widetilde{T}$ and the parent of $\mathbf{n+1}$ has only two children.

FIGURE 31. $u_1 \to 3tu_2v_2$.

(2) The position $\diamond$ is an elder twin leaf $\mathbf{j}$ of T labeled by u_2. Suppose that $\mathbf{k}$ is the twin sibling leaf of $\mathbf{j}$ and $\mathbf{i}$ is the parent of both $\mathbf{j}$ and $\mathbf{k}$, see Fig. 32. It follows the same actions as in Case (1), see Fig. 33. It is straightforward to check the resulting change in weights adheres to the substitution rule $u_2 \to 3tu_2v_1$.

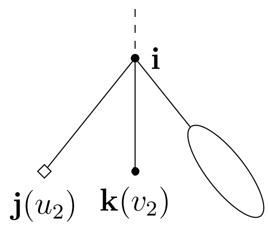
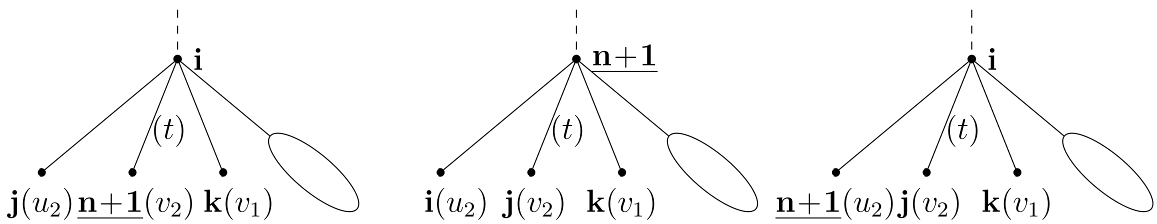


FIGURE 32. The position $\diamond$ is an elder twin leaf of T.



(A): $\mathbf{n+1}$ is a second leaf of $\widetilde{T}$ and the vertex following $\mathbf{n+1}$ is a younger leaf.

(B): $\mathbf{n+1}$ is the parent of an elder twin leaf in $\widetilde{T}$ and $\mathbf{n+1}$ has at least three children where the third child is a younger leaf.

(C): $\mathbf{n+1}$ is an elder twin leaf in $\widetilde{T}$ and the parent of $\mathbf{n+1}$ has at least three children where the third child is a younger leaf.

FIGURE 33. $u_2 \to 3tu_2v_1$.

(3) The position $\diamond$ is an elder non-twin leaf $\mathbf{j}$ of T labeled by u_3. Suppose that $\mathbf{k}$ is the sibling right immediately after $\mathbf{j}$ and $\mathbf{i}$ is the parent of both $\mathbf{j}$ and $\mathbf{k}$, see Fig. 34. It takes the same performances as in Case (1), see Fig. 35. It is clear to find that the weights are satisfies with the substitute rule $u_3 \to 3tu_2v_2$.

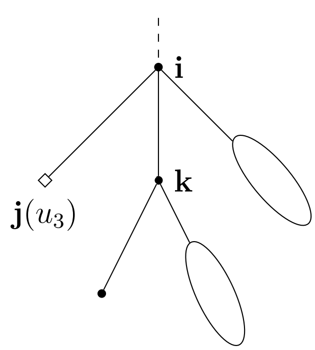
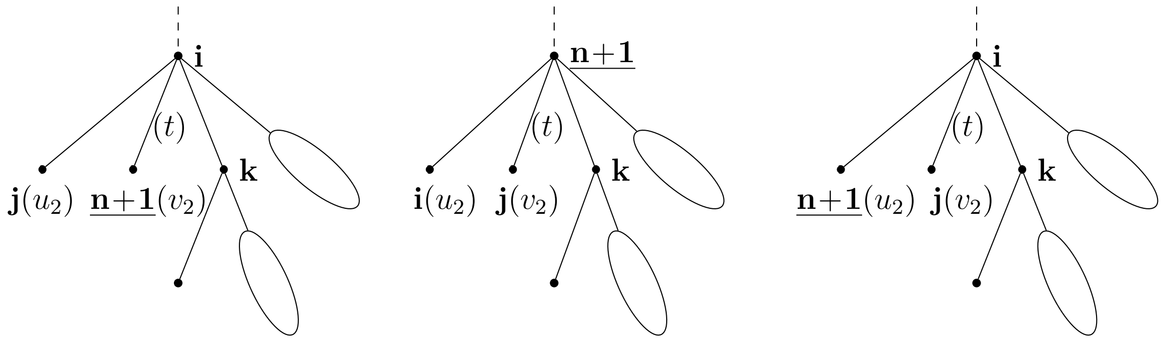


FIGURE 34. The position $\diamond$ is an elder non-twin leaf of T.



(A): $\mathbf{n+1}$ is a second leaf of $\widetilde{T}$ and the vertex adjacent to $\mathbf{n+1}$ is not a younger leaf.

(B): $\mathbf{n+1}$ is the parent of an elder twin leaf in $\widetilde{T}$ and $\mathbf{n+1}$ has at least three children where the third child is not a younger leaf.

(C): $\mathbf{n+1}$ is an elder twin leaf of $\widetilde{T}$ and the parent of $\mathbf{n+1}$ has at least three children where the third child is not a younger leaf.

FIGURE 35. $u_3 \to 3tu_2v_2$.

(4) The position $\diamond$ is a younger leaf $\mathbf{j}$ of T labeled by v_1, as depicted in Fig. 36. It should be noted that the forest A, formed by the subtrees $\mathbf{k_2}, \ldots, \mathbf{k_{t-1}}$ can not be empty, while the forest B, formed by the subtrees $\mathbf{k_{t+1}}, \ldots, \mathbf{k_q}$ may be empty.

(4-i) Insert the vertex $\mathbf{n+1}$ as illustrated in Fig. 37 to form the tree $\widetilde{T}$. A simple check reveals the weights follows the substitution rule $v_1 \to 2u_1$.

(4-ii) Insert the vertex $\mathbf{n+1}$ as illustrated in Fig. 38 to construct the tree $\widetilde{T}$. A simple check reveals the weights follows the substitution rule $v_1 \to 2u_3$.

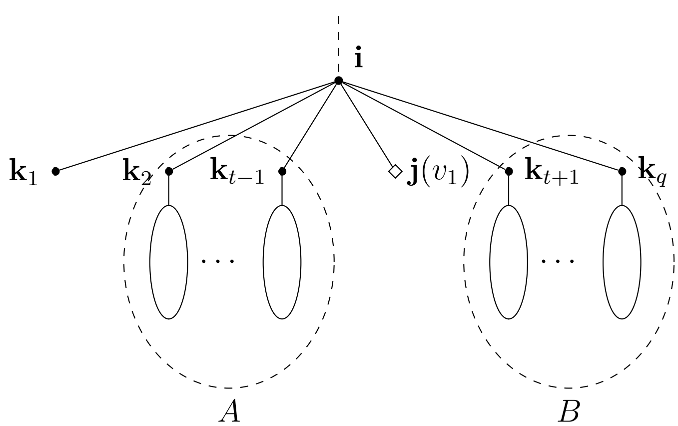
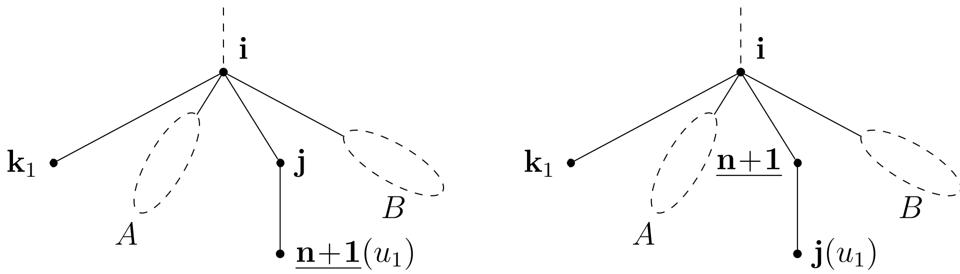


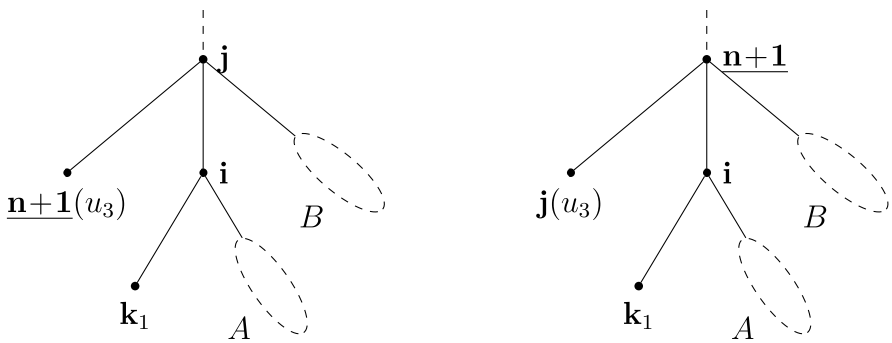
FIGURE 36. The position $\diamond$ is a younger leaf of T.



(A): $\mathbf{n+1}$ is a singleton leaf of $\widetilde{T}$ and the parent of $\mathbf{n+1}$ has at least two elder siblings.

(B): $\mathbf{n+1}$ is the parent of a singleton leaf in $\widetilde{T}$ and $\mathbf{n+1}$ has at least two elder siblings.

FIGURE 37. $v_1 \to 2u_1$.



(A): $\mathbf{n+1}$ is an elder non-twin leaf of $\widetilde{T}$ and the vertex next to $\mathbf{n+1}$ has at least two children.

(B): $\mathbf{n+1}$ is the parent of an elder non-twin leaf in $\widetilde{T}$ and the second child of $\mathbf{n+1}$ has at least two children.

FIGURE 38. $v_1 \to 2u_3$.

(5) The position $\diamond$ is a second leaf $\mathbf{j}$ of T labeled by v_2, as depicted in Fig. 39. We perform the steps shown in Fig. 40 to create the plane tree $\widetilde{T}$. It is easy to check that the change in weights follows the substitution rule $v_2 \to 4u_1u_2^{-1}u_3$.

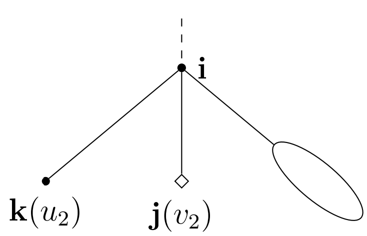
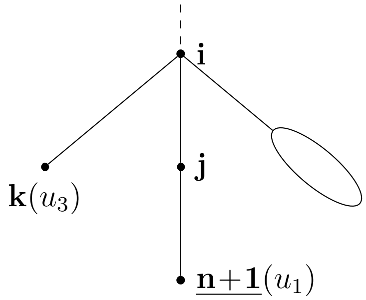
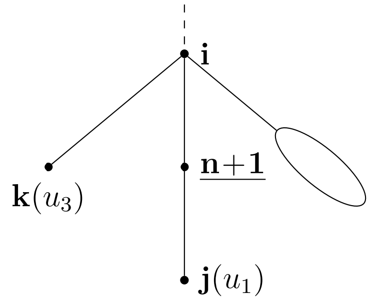


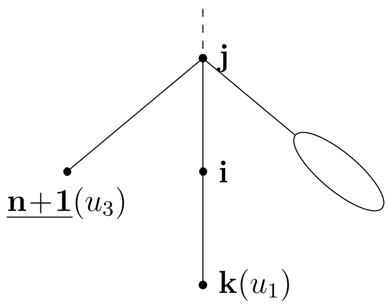
FIGURE 39. The position $\diamond$ is a second leaf of T.



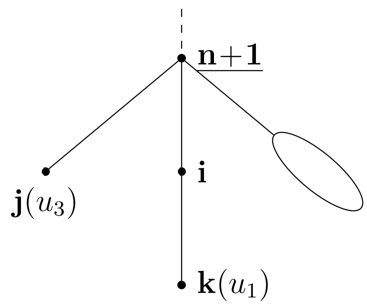
(A): $\mathbf{n+1}$ is a singleton leaf of $\widetilde{T}$ and the parent of $\mathbf{n+1}$ has only elder sibling.



(B): $\mathbf{n+1}$ is the parent of a singleton leaf in $\widetilde{T}$ and $\mathbf{n+1}$ has only elder sibling.



(C): $\mathbf{n+1}$ is an elder non-twin leaf of $\widetilde{T}$ and the vertex next to $\mathbf{n+1}$ has only one child.



(D): $\mathbf{n+1}$ is the parent of an elder non-twin leaf in $\widetilde{T}$ and the second child of $\mathbf{n+1}$ has only one child.

FIGURE 40. $v_2 \to 4u_1u_2^{-1}u_3$.

(6) The position $\diamond$ is a young edge $(\mathbf{i}, \mathbf{j})$ of T with the labeling t, as shown in Fig. 41. Insert the edge $(\mathbf{i}, \mathbf{n+1})$ right immediately after $(\mathbf{i}, \mathbf{j})$ as illustrated in Fig. 42 (A). The change is in accordance with $t \to t^2 v_1$.

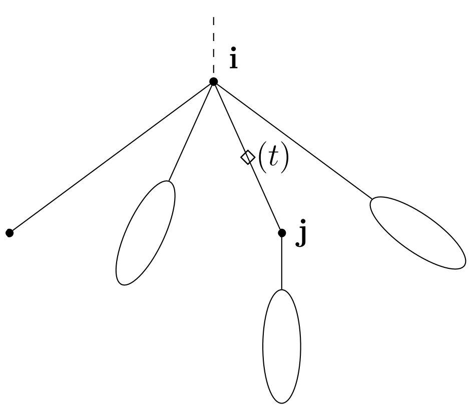
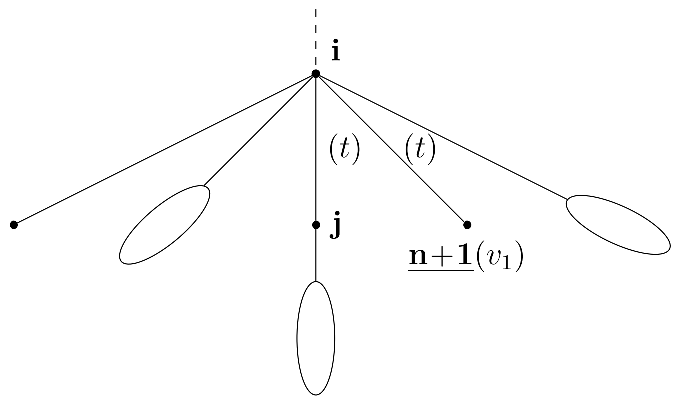


FIGURE 41. The position $\diamond$ is a young edge.



(A): $\mathbf{n+1}$ is a younger leaf of $\widetilde{T}$.

FIGURE 42. $t \to t^2 v_1$.

From the above construction, it can be checked that $\widetilde{T} \in \mathcal{LT}_n$. Moreover,

- $\mathbf{n+1}$ is a singleton leaf of $\widetilde{T}$ in Fig. 37 (A) and Fig. 40 (A).
- $\mathbf{n+1}$ is an elder twin leaf of $\widetilde{T}$ in Fig. 31 (C), Fig. 33 (C) and Fig. 35 (C).
- $\mathbf{n+1}$ is an elder non-twin leaf of $\widetilde{T}$ in Fig. 38 (A) and Fig. 40 (C).
- $\mathbf{n+1}$ is a second leaf of $\widetilde{T}$ in Fig. 31 (A), Fig. 33 (A) and Fig. 35 (A).
- $\mathbf{n+1}$ is a younger leaf of $\widetilde{T}$ in Fig. 42 (A).
- $\mathbf{n+1}$ is a parent of a singleton leaf of $\widetilde{T}$ in Fig. 37 (B) and Fig. 40 (B).
- $\mathbf{n+1}$ is a parent of an elder twin leaf of $\widetilde{T}$ in Fig. 31 (B), Fig. 33 (B) and Fig. 35 (B).
- $\mathbf{n+1}$ is a parent of an elder non-twin leaf of $\widetilde{T}$ in Fig. 38 (B) and Fig. 40 (D).

Hence, this construction is reversible according to the position of the vertex $\mathbf{n+1}$ in $\widetilde{T}$. Therefore, we could generate all tip-augmented plane trees in $\mathcal{LT}_n$ from those plane trees in $\mathcal{LT}_{n-1}$ based on the above construction. Moreover, the change of weights between tip-augmented plane trees in $\mathcal{LT}_n$ and tip-augmented plane trees in $\mathcal{LT}_{n-1}$ aligns with the substitution rule given in the grammar (4.1). Thus we show that (4.4) is valid, and so (4.2) is also valid for $n-1$. This completes the proof. $\square$

4.2. **The grammatical derivation for Theorem 4.2.** Considering the formal derivative D with respect to the grammar M as given in (4.1), we observe that

$$D(t^{-1}) = -v_1,$$

$$D^2(t^{-1}) = -2(u_1 + u_3),$$

$$D^3(t^{-1}) = -12tu_2v_2 = -2D(2u_3).$$

Thus, by applying Theorem 4.1, we conclude that for $n \geq 3$,

$$D^n(t^{-1}) = -2D^{n-2}(2u_3) = -2n!M_{n-2}(u_1,u_2,u_3;v_1,v_2;t).$$

Under the assumption $M_0(u_1,u_2,u_3;v_1,v_2;t) = u_3$, we derive that

$$\begin{aligned}
\mathrm{Gen}(t^{-1};q) &= t^{-1} - v_1q - (u_1+u_3)q^2 + \sum_{n\geq 3} D^n(t^{-1})\frac{q^n}{n!} \\
&= t^{-1} - v_1q - (u_1+u_3)q^2 - 2\sum_{n\geq 3} n!M_{n-2}(u_1,u_2,u_3;v_1,v_2;t)\frac{q^n}{n!} \\
&= t^{-1} - v_1q - (u_1+u_3)q^2 - 2q^2\sum_{n\geq 1} M_n(u_1,u_2,u_3;v_1,v_2;t)q^n \\
&= t^{-1} - v_1q - (u_1+u_3)q^2 - 2q^2\sum_{n\geq 0} M_n(u_1,u_2,u_3;v_1,v_2;t)q^n + 2u_3q^2 \\
&= t^{-1} - v_1q + (u_3-u_1)q^2 - 2q^2\sum_{n\geq 0} M_n(u_1,u_2,u_3;v_1,v_2;t)q^n. \qquad (4.5)
\end{aligned}$$

Since

$$\mathrm{Gen}(t^{-2};q) = \mathrm{Gen}^2(t^{-1};q), \qquad (4.6)$$

it suffices to consider $\mathrm{Gen}(t^{-2};q)$. Observe that

$$\begin{aligned}
D(t^{-2}) &= -2t^{-1}v_1, \\
D^2(t^{-2}) &= 2v_1^2 - 4t^{-1}(u_1+u_3), \\
D^3(t^{-2}) &= 12v_1(u_1+u_3) - 24u_2v_2, \\
D^4(t^{-2}) &= 24(u_1-u_3)^2, \\
D^5(t^{-2}) &= 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathrm{Gen}(t^{-2};q) &= \sum_{n\geq 0} D^n(t^{-2})\frac{q^n}{n!} \\
&= t^{-2} - 2t^{-1}v_1q + (v_1^2 - 2t^{-1}(u_1+u_3))q^2 + (2u_1v_1 + 2u_3v_1 - 4u_2v_2)q^3 \\
&\quad + (u_1-u_3)^2q^4. \qquad (4.7)
\end{aligned}$$

By setting $q = 0$ in (4.6), we infer from (4.5) and (4.7) that

$$\mathrm{Gen}(t^{-1};q) = \sqrt{\mathrm{Gen}(t^{-2};q)}. \qquad (4.8)$$

Hence we arrive at the generating function for $M_n(u_1,u_2,u_3;v_1,v_2;t)$ stated in Theorem 4.2 by utilizing (4.5), (4.7) and (4.8). This completes the proof. $\square$

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